2.1 Derivative Function

A Derivative Function
Given a function $y = f(x)$, the *derivative function* of f is a *new function* called f' (*f prime*), defined at x by:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

B Differentiability
A function $y = f(x)$ is called *differentiable* at x if $f'(x)$ exists.
A function $y = f(x)$ is differentiable over an open interval (a,b) if the function is differentiable at every number in that interval.
Note: The domain of derivative function f' is a subset of the domain of the original function f: $D_{f'} \subset D_f$. So a function is defined over D_f but is differentiable over $D_{f'}$.

C Interpretations of Derivative Function
1. The *slope of the tangent line* to the graph of $y = f(x)$ at the point $P(a, f(a))$ is given by $m = f'(a)$.
2. The *instantaneous rate of change* in the variable y with respect to the variable x, where $y = f(x)$, at $x = a$ is given by:
$IRC = f'(a)$.

D Notations and Reading
$y' = f'(x)$ [Lagrange Notation]
Read: "y prime" or "f prime at x"

$\frac{dy}{dx} = \frac{d}{dx}f(x)$ [Leibnitz Notation]
Read: "dee y by dee x"

$f'(a) = \left.\frac{dy}{dx}\right|_{x=a}$
Read: "f prime at a, dee y by dee x at x equals a"

E First Principles
Differentiation is the process to find the derivative function for a given function.

First Principles is the process of differentiation by computing the limit:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Ex 1. Use first principles to differentiate each function.
a) $f(x) = 2x - x^3$

$f(x+h) = 2(x+h) - (x+h)^3$

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{2(x+h)-(x+h)^3-[2x-x^3]}{h}$

$= \lim_{h\to 0}\frac{2(x+h)-2x}{h} - \lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$

$= \lim_{h\to 0}2 - \lim_{h\to 0}\frac{h[(x+h)^2+(x+h)x+x^2]}{h} = 2-3x^2$

$\therefore f'(x) = 2-3x^2$

b) $f(x) = \frac{-3}{x^2}$

$f(x+h) = \frac{-3}{(x+h)^2}$

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{-3}{(x+h)^2}-\frac{-3}{x^2}}{h}$

$= -3\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} = -3\lim_{h\to 0}\frac{x^2-(x+h)^2}{h(x+h)^2x^2}$

$= -3\lim_{h\to 0}\frac{x^2-[x^2+2xh+h^2]}{h(x+h)^2x^2} = -3\lim_{h\to 0}\frac{-2x-h}{(x+h)^2x^2}$

$= -3\frac{-2x}{x^2x^2} = \frac{6}{x^3}$

$\therefore f'(x) = \frac{6}{x^3}$

c) $f(x) = \sqrt{ax+b}$

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{a(x+h)+b}-\sqrt{ax+b}}{h}$

$= \lim_{h\to 0}\frac{\sqrt{a(x+h)+b}-\sqrt{ax+b}}{h}\frac{\sqrt{a(x+h)+b}+\sqrt{ax+b}}{\sqrt{a(x+h)+b}+\sqrt{ax+b}}$

$= \lim_{h\to 0}\frac{a(x+h)+b-(ax+b)}{h(\sqrt{a(x+h)+b}+\sqrt{ax+b})}$

$= \lim_{h\to 0}\frac{ah}{h(\sqrt{a(x+h)+b}+\sqrt{ax+b})} = \lim_{h\to 0}\frac{a}{\sqrt{a(x+h)+b}+\sqrt{ax+b}}$

$= \frac{a}{\sqrt{a(x+0)+b}+\sqrt{ax+b}} = \frac{a}{2\sqrt{ax+b}}$

$\therefore f'(x) = \frac{a}{2\sqrt{ax+b}}$

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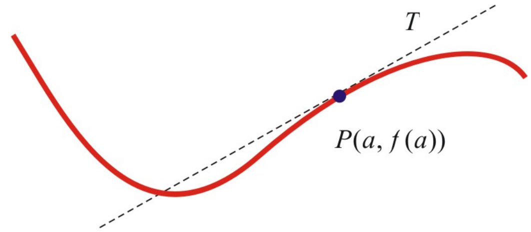
G Non-Differentiability
A function is *not differentiable* at $x = a$ if $f'(a)$ *does not exist*.

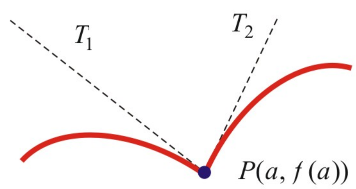
Notes:
1. If a function f is *not continuous* at $x = a$ then the function f is *not differentiable* at $x = a$.
2. If a function f is *continuous* at $x = a$ then the function f *may be or not* differentiable at $x = a$.

H Differentiability Point
If the function $y = f(x)$ is *differentiable* at $x = a$ then the tangent line at $P(a, f(a))$ is *unique* and *not vertical* (the slope of the tangent line is not ∞ or $-\infty$).



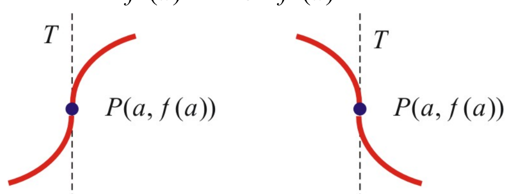
I Corner Point
$P(a, f(a))$ is a *corner point* if there are *two* distinct tangent lines at P, one for the left-hand branch and one for the right-hand branch. For example:

$$f(x) = \begin{cases} f_1(x), & x < a \\ f_2(x), & x > a \end{cases} \quad \text{and} \quad f_1'(a) \neq f_2'(a)$$



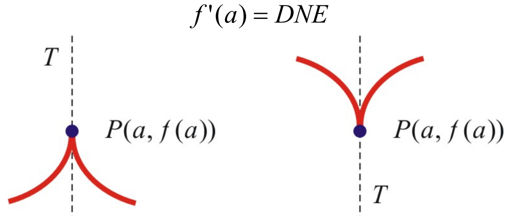
J Infinite Slope Point
$P(a, f(a))$ is a *infinite slope point* if the tangent line at P is vertical and the function is increasing or decreasing in the neighborhood at the of the point P.

$$f'(a) = \infty \text{ or } f'(a) = -\infty$$

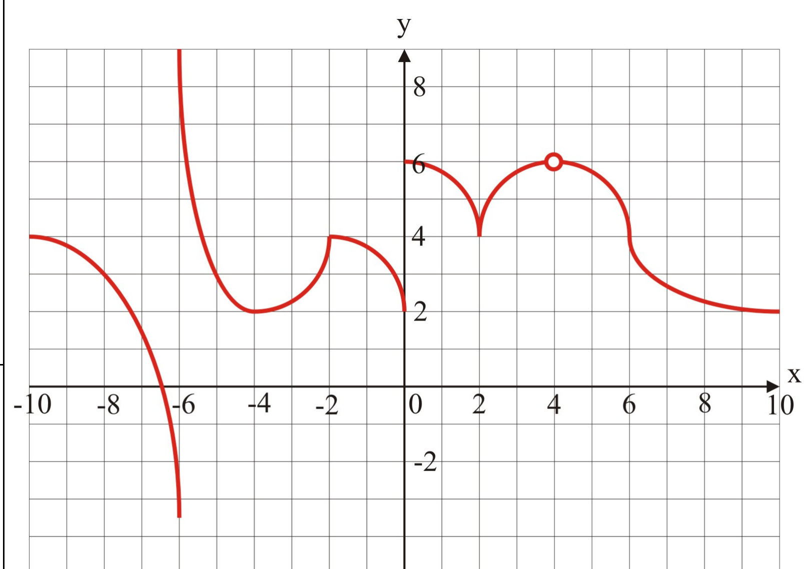


K Cusp Point
$P(a, f(a))$ is a *cusp point* if the tangent line at P is vertical and the function is increasing on one side of the point P and decreasing on the other side.

$$f'(a) = DNE$$



Ex 2. Find the numbers x where the function $y = f(x)$ (see the graph below) is not differentiable and explain why.



The function $y = f(x)$ is not differentiable at:
$x = -6$ (infinite break)
$x = -2$ ($P(-2,4)$ is a corner point)
$x = 0$ (jump discontinuity)
$x = 2$ ($P(2,4)$ is a cusp point)
$x = 4$ (removable discontinuity)
$x = 6$ ($P(6,4)$ is an infinite slope point)

Reading: Nelson Textbook, Pages 65-72
Homework: Nelson Textbook: Page 73 #1, 6, 7b, 9, 14, 16, 19

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