2.5 Chain Rule

A Composition of functions

If $u = g(x)$ and $v = f(u)$ then:

$$x \underset{u=g(x)}{\rightarrow} u \underset{v=f(u)}{\rightarrow} v$$

and

$$v = f(u) = f(g(x)) = (f \circ g)(x)$$

B Chain Rule (Leibniz Notation)

$$\Delta x \underset{u=g(x)}{\rightarrow} \Delta u \underset{v=f(u)}{\rightarrow} \Delta v$$

and

$$\frac{\Delta v}{\Delta x} = \frac{\Delta v}{\Delta u}\frac{\Delta u}{\Delta x} \quad \rightarrow \quad \frac{dv}{dx} = \frac{dv}{du}\frac{du}{dx}$$

Therefore:

$$\frac{dv}{dx} = \frac{dv}{du}\frac{du}{dx}$$

Ex 1. Consider $u = x^2 - x$ and $v = \sqrt{u}$. Find $\frac{dv}{dx}$.

$$\frac{dv}{dx} = \frac{dv}{du}\frac{du}{dx}$$

$$\frac{dv}{du} = \frac{1}{2\sqrt{u}}, \quad \frac{du}{dx} = 2x - 1 \quad \Rightarrow \quad \frac{dv}{dx} = \frac{1}{2\sqrt{u}}(2x-1)$$

$$\therefore \frac{dv}{dx} = \frac{1}{2\sqrt{x^2 - x}}(2x-1)$$

Ex 2. Consider $u = \sqrt{x}$ and $v = \frac{u}{u-1}$. Find $\left.\frac{dv}{dx}\right|_{x=4}$.

$$\frac{dv}{dx} = \frac{dv}{du}\frac{du}{dx}$$

$$\frac{dv}{du} = \frac{1(u-1) - u}{(u-1)^2} = \frac{-1}{(u-1)^2}, \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$\frac{dv}{dx} = \frac{-1}{(u-1)^2}\frac{1}{2\sqrt{x}}$$

If $x = 4$ then $u = \sqrt{4} = 2$. So:

$$\left.\frac{dv}{dx}\right|_{x=4} = \frac{-1}{(2-1)^2}\frac{1}{2\sqrt{4}} = -\frac{1}{4}$$

$$\therefore \left.\frac{dv}{dx}\right|_{x=4} = -\frac{1}{4}$$

C Composition of three functions

$$x \underset{u=h(x)}{\rightarrow} u \underset{v=g(u)}{\rightarrow} v \underset{w=f(v)}{\rightarrow} w$$

$$\frac{dw}{dx} = \frac{dw}{dv}\frac{dv}{du}\frac{du}{dx}$$

Ex 3. Consider $u = x^2$, $v = \frac{1}{u+1}$, and $w = \sqrt{v}$. Find $\frac{dw}{dx}$.

$$\frac{dw}{dx} = \frac{dw}{dv}\frac{dv}{du}\frac{du}{dx}$$

$$\frac{dw}{dv} = \frac{1}{2\sqrt{v}}, \quad \frac{dv}{du} = -\frac{1}{(u+1)^2}, \quad \frac{du}{dx} = 2x$$

$$\frac{dw}{dx} = \frac{1}{2\sqrt{v}}\left(-\frac{1}{(u+1)^2}\right)(2x) = \frac{1}{2\sqrt{\frac{1}{x^2+1}}}\left(-\frac{1}{(x^2+1)^2}\right)(2x)$$

$$\therefore \frac{dw}{dx} = -\frac{x\sqrt{x^2+1}}{(x^2+1)^2}$$

D Chain Rule (Lagrange Notation)

$$v = f(u) = f(g(x)) = (f \circ g)(x)$$
$$\frac{dv}{dx} \to [f(g(x))]'$$
$$\frac{dv}{du} \to f'(u) = f'(g(x)$$
$$\frac{du}{dx} \to g'(x)$$
$$\frac{dv}{dx} = \frac{dv}{du}\frac{du}{dx} \quad \to \quad [f(g(x))]' = f'(g(x))g'(x)$$

If g is differentiable at x and f is differentiable at $f(x)$ then the composition $(f \circ g)(x) = f(g(x))$ is differentiable at x and

$$(f \circ g)'(x) = [f(g(x))]' = f'(g(x))g'(x)$$

So, the derivative of $f(g(x))$ is the derivative of the *outside* function f evaluated of the *inside* function g times the derivative of the inside function g.

Ex 4. Differentiate $f(x) = (x^3 - 2x^2 + x)^5$.

$$f'(x) = [(x^3 - 2x^2 + x)^5]'$$
$$= 5(x^3 - 2x^2 + x)^{5-1}(x^3 - 2x^2 + x)'$$
$$= 5(x^3 - 2x^2 + x)^4(3x^2 - 4x + 1)$$
$$\therefore f'(x) = 5(x^3 - 2x^2 + x)^4(3x^2 - 4x + 1)$$

Ex 5. Differentiate $f(x) = \sqrt[3]{x^2 - \sqrt{x}}$.

$$f'(x) = \left[(x^2 - \sqrt{x})^{\frac{1}{3}}\right]' = \frac{1}{3}(x^2 - \sqrt{x})^{\frac{1}{3}-1}(x^2 - \sqrt{x})'$$
$$= \frac{1}{3\sqrt[3]{(x^2 - \sqrt{x})^2}}\left(2x - \frac{1}{2\sqrt{x}}\right)$$
$$\therefore f'(x) = \frac{2x - \frac{1}{2\sqrt{x}}}{3\sqrt[3]{(x^2 - \sqrt{x})^2}}$$

Ex 6. Differentiate $f(x) = x^2\sqrt{\frac{x+1}{x^2+1}}$.

$$f'(x) = (x^2)'\sqrt{\frac{x+1}{x^2+1}} + x^2\left(\sqrt{\frac{x+1}{x^2+1}}\right)'$$
$$= 2x\sqrt{\frac{x+1}{x^2+1}} + x^2\left[\left(\frac{x+1}{x^2+1}\right)^{\frac{1}{2}}\right]'$$
$$= 2x\sqrt{\frac{x+1}{x^2+1}} + x^2\frac{1}{2}\left(\frac{x+1}{x^2+1}\right)^{\frac{1}{2}-1}\left(\frac{x+1}{x^2+1}\right)'$$
$$= 2x\sqrt{\frac{x+1}{x^2+1}} + x^2\frac{1}{2\sqrt{\frac{x+1}{x^2+1}}}\frac{1(x^2+1) - (x+1)(2x)}{(x^2+1)^2}$$
$$\therefore f'(x) = 2x\sqrt{\frac{x+1}{x^2+1}} + x^2\frac{1 - 2x - x^2}{2(x^2+1)^2\sqrt{\frac{x+1}{x^2+1}}}$$

Ex 7. Differentiate $f(x) = x + \sqrt{x^2 + \sqrt{x^3 + \sqrt{x^4 + 1}}}$.

$$f'(x) = 1 + \frac{2x + \frac{3x^2 + \frac{4x^3}{2\sqrt{x^4+1}}}{2\sqrt{x^3 + \sqrt{x^4+1}}}}{2\sqrt{x^2 + \sqrt{x^3 + \sqrt{x^4+1}}}}$$

Reading: Nelson Textbook, Pages 94-95
Homework: Nelson Textbook: Page 95 #4f, 5b, 8, 9a, 12, 14, 15, 16

