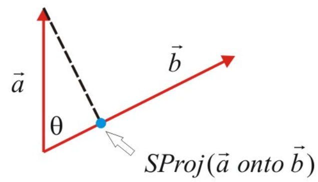
7.5 Scalar and Vector Projections

A Scalar Projection

The *scalar projection* of the vector $\vec{a}$ onto the vector $\vec{b}$ is a *scalar* defined as:

$$SProj(\vec{a}\ onto\ \vec{b}) = \|\vec{a}\|\cos\theta \quad \text{where } \theta = \angle(\vec{a},\vec{b})$$



Ex 1. Given two vectors with the magnitudes $\|\vec{a}\| = 10$ and $\|\vec{b}\| = 16$ respectively, and the angle between them equal to $\theta = 120°$, find the scalar projection

a) of the vector $\vec{a}$ onto the vector $\vec{b}$

$$SProj(\vec{a}\ onto\ \vec{b}) = \|\vec{a}\|\cos\theta = 10\cos 120° = -5$$

b) of the vector $\vec{b}$ onto the vector $\vec{a}$

$$SProj(\vec{b}\ onto\ \vec{a}) = \|\vec{b}\|\cos\theta = 16\cos 120° = -8$$

B Special Cases

Consider two vectors $\vec{a}$ and $\vec{b}$.

a) If $\vec{a} \uparrow\uparrow \vec{b}$ ($\cos\theta = 1$), then $SProj(\vec{a}\ onto\ \vec{b}) = \|\vec{a}\|$

b) If $\vec{a} \uparrow\downarrow \vec{b}$ ($\cos\theta = -1$), then $SProj(\vec{a}\ onto\ \vec{b}) = -\|\vec{a}\|$

c) If $\vec{a} \perp \vec{b}$ then $SProj(\vec{a}\ onto\ \vec{b}) = 0$

Ex 2. Find the scalar projection of the vector $\vec{a}$ onto:

a) itself

$$SProj(\vec{a}\ onto\ \vec{a}) = \|\vec{a}\|$$

b) the opposite vector $-\vec{a}$

$$SProj(\vec{a}\ onto - \vec{a}) = -\|\vec{a}\|$$

C Dot Product and Scalar Projection

Recall that the *dot product* of the vectors $\vec{a}$ and $\vec{b}$ is defined as:

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$$

So, the *scalar projection* of the vector $\vec{a}$ onto the vector $\vec{b}$ can be written as:

$$SProj(\vec{a}\ onto\ \vec{b}) = \|\vec{a}\|\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}$$

Note:
For a Cartesian (Rectangular) coordinate system, the *scalar components* a_x, a_y, and a_z of a vector $\vec{a} = (a_x, a_y, a_z)$ are the *scalar projections* of the vector $\vec{a}$ onto the unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$.

Proof:

$$SProj(\vec{a}\ onto\ \vec{i}) = \frac{\vec{a}\cdot\vec{i}}{\|\vec{i}\|} = \frac{(a_x, a_y, a_z)\cdot(1,0,0)}{1} = a_x$$

Ex 3. Given the vector $\vec{a} = (2,-3,4)$, find the scalar projection:

a) of $\vec{a}$ onto the unit vector $\vec{i}$

$$SProj(\vec{a}\ onto\ \vec{i}) = \frac{\vec{a}\cdot\vec{i}}{\|\vec{i}\|} = 2$$

b) of $\vec{a}$ onto the vector $\vec{i} - \vec{j}$

$$SProj(\vec{a}\ onto\ \vec{i} - \vec{j}) = \frac{\vec{a}\cdot(\vec{i} - \vec{j})}{\|\vec{i} - \vec{j}\|} = \frac{(2,-3,4)\cdot(1,-1,0)}{\sqrt{2}} = \frac{5}{\sqrt{2}}$$

c) of $\vec{a}$ onto the vector $\vec{b} = -\vec{i} + 2\vec{j} + \vec{k}$

$$SProj(\vec{a}\ onto\ \vec{b}) = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|} = \frac{(2,-3,4)\cdot(-1,2,1)}{\sqrt{6}} = \frac{-2-6+4}{\sqrt{1+4+1}} = \frac{-4}{\sqrt{6}}$$

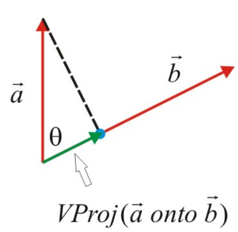
d) of the unit vector $\vec{i}$ onto the vector $\vec{a}$

$$SProj(\vec{i}\ onto\ \vec{a}) = \frac{\vec{i}\cdot\vec{a}}{\|\vec{a}\|} = \frac{(1,0,0)\cdot(2,-3,4)}{\sqrt{4+9+16}} = \frac{2}{\sqrt{29}}$$

D Vector Projection

The *vector projection* of the vector $\vec{a}$ onto the vector $\vec{b}$ is a vector defined as:

$$VProj(\vec{a}\ onto\ \vec{b}) = \|\vec{a}\|\cos\theta\frac{\vec{b}}{\|\vec{b}\|}$$



E Dot Product and Vector Projection

The *vector projection* of the vector $\vec{a}$ onto the vector $\vec{b}$ can be written using the dot product as:

$$VProj(\vec{a}\ onto\ \vec{b}) = \frac{(\vec{a}\cdot\vec{b})\vec{b}}{\|\vec{b}\|^2}$$

Note:
For a Cartesian (Rectangular) coordinate system, the *vector components* $\vec{a}_x = a_x\vec{i}$, $\vec{a}_y = a_y\vec{j}$, and $\vec{a}_z = a_z\vec{k}$ of a vector $\vec{a} = (a_x, a_y, a_z)$ are the *vector projections* of the vector $\vec{a}$ onto the unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$.

Ex 4. Given two vectors $\vec{a}=(0,1,-2)$ and $\vec{b}=(-1,0,3)$, find:

a) the vector projection of the vector $\vec{a}$ onto the vector $\vec{b}$

$$VProj(\vec{a}\ onto\ \vec{b})=\frac{(\vec{a}\cdot\vec{b})\vec{b}}{\|\vec{b}\|^2}=\frac{(-6)(-1,0,3)}{1+0+9}=\frac{(6,0,-18)}{10}=\left(\frac{3}{5},0,\frac{-9}{5}\right)$$

b) the vector projection of the vector $\vec{b}$ onto the vector $\vec{a}$

$$VProj(\vec{b}\ onto\ \vec{a})=\frac{(\vec{b}\cdot\vec{a})\vec{a}}{\|\vec{a}\|^2}=\frac{(-6)(0,1,-2)}{0+1+4}=\frac{(0,-6,12)}{5}=\left(0,\frac{-6}{5},\frac{12}{5}\right)$$

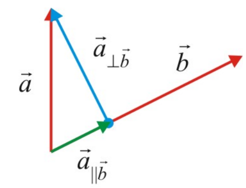
c) the vector projection of the vector $\vec{a}$ onto the unit vector $\vec{k}$

$$VProj(\vec{a}\ onto\ \vec{k})=\frac{(\vec{a}\cdot\vec{k})\vec{k}}{\|\vec{k}\|^2}=\frac{(-2)(0,0,1)}{1}=(0,0,-2)$$

d) the vector projection of the vector $\vec{i}$ onto the vector $\vec{a}$

$$VProj(\vec{i}\ onto\ \vec{a})=\frac{(\vec{i}\cdot\vec{a})\vec{a}}{\|\vec{a}\|^2}=\frac{(0)(0,1,-2)}{0+1+4}=(0,0,0)=\vec{0}$$

Ex 5. Find an expression using the dot product of the vector components of the vector $\vec{a}$ along to the vector $\vec{b}$ and along to a direction perpendicular to the direction of the vector $\vec{b}$ but in the same plan containing the vectors $\vec{a}$ and $\vec{b}$.



Let $\vec{a}_{\|\vec{b}}$ be the vector component parallel to $\vec{b}$. Then:

$$\vec{a}_{\|\vec{b}}=VProj(\vec{a}\ onto\ \vec{b})=\frac{(\vec{a}\cdot\vec{b})\vec{b}}{\|\vec{b}\|^2}$$

Let $\vec{a}_{\perp\vec{b}}$ be the vector component perpendicular to $\vec{b}$. Then:

$$\vec{a}_{\perp\vec{b}}=\vec{a}-\vec{a}_{\|\vec{b}}=\vec{a}-\frac{(\vec{a}\cdot\vec{b})\vec{b}}{\|\vec{b}\|^2}$$

Reading: Nelson Textbook, Pages 390-398
Homework: Nelson Textbook: Page 399 # 6, 11, 13, 14

