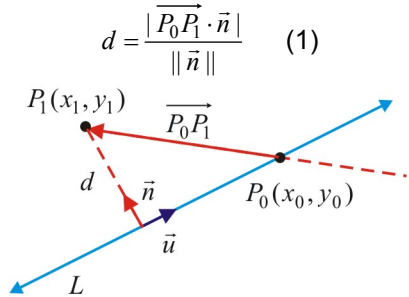


9.5 Distance from a Point to a Line

A Distance from a Point to a Line in \mathbb{R}^2
 Let $L: Ax + By + C = 0$ be a line in \mathbb{R}^2 , $P_1(x_1, y_1)$ be a *generic point* on the xy -plane and $P_0(x_0, y_0)$ be a *specific point* on this line, so: $Ax_0 + By_0 + C = 0$.
 The *distance* d between the point $P_1(x_1, y_1)$ to the line L is given by (*scalar projection* of $\overrightarrow{P_0P_1}$ onto the normal vector \vec{n}):



$$d = \frac{|\overrightarrow{P_0P_1} \cdot \vec{n}|}{\|\vec{n}\|} \quad (1)$$

Using $\vec{n} = (A, B)$, $\|\vec{n}\| = \sqrt{A^2 + B^2}$ and:
 $\overrightarrow{P_0P_1} \cdot \vec{n} = (x_1 - x_0, y_1 - y_0) \cdot (A, B)$
 $= A(x_1 - x_0) + B(y_1 - y_0) = Ax_1 + By_1 - Ax_0 - By_0$
 $= Ax_1 + By_1 + C$

the formula (1) may be written as:

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \quad (2)$$

Ex 1. Find the distance between the point $P_1(3, -1)$ and the line $L: -2x + 3y + 6 = 0$.

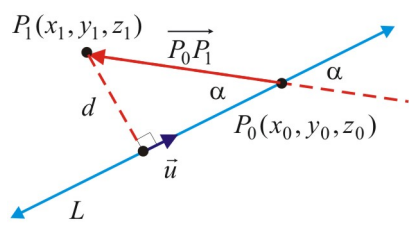
Method #1 is based on formula (1).
 If $x = 0$ then $y = -2$. So $P_0(0, -2) \in L$.

$$\begin{aligned} \overrightarrow{P_0P_1} &= (3, -1) - (0, -2) = (3, 1) \\ \vec{n} &= (-2, 3) \\ d &= \frac{|\overrightarrow{P_0P_1} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|-6 + 3|}{\sqrt{4 + 9}} = \frac{3}{\sqrt{13}} \end{aligned}$$

Method #2 is based on formula (2).

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|-2(3) + 3(-1) + 6|}{\sqrt{(-2)^2 + 3^2}} = \frac{3}{\sqrt{13}}$$

B Distance from a Point to a Line in \mathbb{R}^3
 Let $L: \vec{r} = \vec{r}_0 + t\vec{u}$, $t \in \mathbb{R}$ be a line defined by its vector equation and $P_0(x_0, y_0, z_0)$ be a specific point on this line.



The *distance* d from a point $P_1(x_1, y_1, z_1)$ to the line L may be found using:

$$d = \|\overrightarrow{P_0P_1}\| \sin \alpha \quad (3)$$

Because $\|\overrightarrow{P_0P_1} \times \vec{u}\| = \|\overrightarrow{P_0P_1}\| \|\vec{u}\| \sin \alpha$, the formula (3) may be written as:

$$d = \frac{\|\overrightarrow{P_0P_1} \times \vec{u}\|}{\|\vec{u}\|} \quad (4)$$

Note. The formula (4) may be applied also in \mathbb{R}^2 by considering the third component $z = 0$.

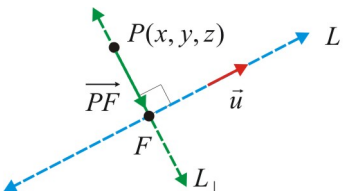
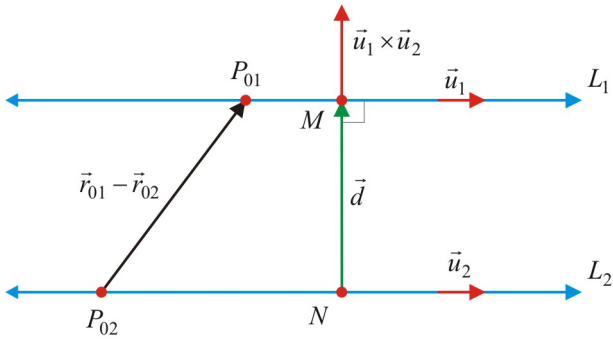
Ex 2. For each case, find the distance from the given point to the given line.

a) $L: \vec{r} = (0, 1, 2) + t(-1, 2, 0)$, $t \in \mathbb{R}$; $P_1(1, 2, -1)$

$$\begin{aligned} P_0(0, 1, 2), P_1(1, 2, -1) &\Rightarrow \overrightarrow{P_0P_1} = (1, 1, -3) \\ \vec{u} &= (-1, 2, 0) \\ \overrightarrow{P_0P_1} \times \vec{u} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -3 \\ -1 & 2 & 0 \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} \\ 1 & 1 \\ -1 & 2 \end{vmatrix} = (6, 3, 3) \\ d &= \frac{\|\overrightarrow{P_0P_1} \times \vec{u}\|}{\|\vec{u}\|} = \frac{\sqrt{36 + 9 + 9}}{\sqrt{1 + 4}} = \sqrt{5} \end{aligned}$$

b) $L: \begin{cases} x = -1 + 2t \\ y = -3t \\ z = 2 \end{cases}$, $t \in \mathbb{R}$; $O(0, 0, 0)$

$$\begin{aligned} P_0(-1, 0, 2), O(0, 0, 0) &\Rightarrow \overrightarrow{P_0O} = (1, 0, -2); \vec{u} = (2, -3, 0) \\ \overrightarrow{P_0O} \times \vec{u} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ 2 & -3 & 0 \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} \\ 1 & 0 \\ 2 & -3 \end{vmatrix} = (-6, -4, -3) \\ d &= \frac{\|\overrightarrow{P_0O} \times \vec{u}\|}{\|\vec{u}\|} = \frac{\sqrt{36 + 16 + 9}}{\sqrt{4 + 9}} = \sqrt{13} \end{aligned}$$

<p>C Distance between two Parallel Lines To find the <i>distance</i> between two parallel lines: a) Find a <i>specific point</i> on one of these lines. b) Find the distance from that specific point to the other line using one of the relations above.</p>	<p>Ex 3. Find the distance between the parallel lines: $L_1 : 2x - 3y + 6 = 0$ and $L_2 : -4x + 6y - 3 = 0$. $P_1(0,2) \in L_1$ $\therefore d = \frac{ Ax_1 + By_1 + C }{\sqrt{A^2 + B^2}} = \frac{ -4(0) + 6(2) - 3 }{\sqrt{(-4)^2 + 6^2}} = \frac{9}{\sqrt{52}}$</p>
<p>D Perpendicular Line from a Point to a Line Let $L : \vec{r} = \vec{r}_0 + t\vec{u}$, $t \in R$ be a line defined by its vector equation and $P(x, y, z)$ be a generic point in R^3. The line perpendicular to the line L that passes through the point P is called the <i>perpendicular line</i> and intersects the line L at a point F called the <i>foot</i> of the perpendicular line.</p>  <p>The foot F of the perpendicular line may be found from the equation (because $\vec{PF} \perp \vec{u}$): $\vec{PF} \cdot \vec{u} = 0$ A <i>vector equation</i> of the perpendicular line is: $\vec{r} = \vec{OP} + s\vec{PF}, \quad s \in R$</p>	<p>Ex 4. Consider the line $L : \vec{r} = (-2, 3, 1) + t(1, -3, 2)$, $t \in R$ and the point $P(3, -4, 2)$. a) Find the foot F of the perpendicular line L_\perp from the point P to the line L. $P(3, -4, 2)$; $\vec{u} = (1, -3, 2)$ $F(-2 + t, 3 - 3t, 1 + 2t)$ $\vec{PF} = (-5 + t, 7 - 3t, -1 + 2t)$ $\vec{PF} \cdot \vec{u} = 0 \Rightarrow -5 + t - 3(7 - 3t) + 2(-1 + 2t) = 0$ $-5 + t - 21 + 9t - 2 + 4t = 0 \Rightarrow 14t = 28 \Rightarrow t = 2$ $F(-2 + 2, 3 - 3(2), 1 + 2(2)) \Rightarrow \therefore F(0, -3, 5)$ b) Find the equation of the perpendicular line L_\perp from the point P to the line L. $\vec{PF} = (0, -3, 5) - (3, -4, 2) = (-3, 1, 3)$ $\therefore L_\perp : \vec{r} = (3, -4, 2) + s(-3, 1, 3), \quad s \in R$ c) Find the distance from the point P to the line L. $\therefore d = \ \vec{PF}\ = \sqrt{9 + 1 + 9} = \sqrt{19}$</p>
<p>E Shortest Distance between two Skew Lines Two skew lines lie into <i>two parallel planes</i>. The vector $\vec{u}_1 \times \vec{u}_2$ is <i>perpendicular</i> to both lines and therefore perpendicular to parallel plane the lines lie on. See the diagram below:</p>  <p>The <i>shortest distance</i> between two skew lines $L_1 : \vec{r} = \vec{r}_{01} + t\vec{u}_1$, $t \in R$ and $L_2 : \vec{r} = \vec{r}_{02} + s\vec{u}_2$, $s \in R$ is given by the <i>scalar projection</i> of the vector $\vec{r}_{01} - \vec{r}_{02}$ onto the vector $\vec{u}_1 \times \vec{u}_2$:</p> $d = \frac{ (\vec{r}_{01} - \vec{r}_{02}) \cdot (\vec{u}_1 \times \vec{u}_2) }{\ \vec{u}_1 \times \vec{u}_2\ }$	<p>Ex 5. Find the shortest distance between the lines $L_1 : \vec{r} = (2, 0, 0) + t(1, 1, 0)$, $t \in R$ and $L_2 : \vec{r} = (1, 1, 1) + s(0, -1, 2)$, $s \in R$.</p> $\vec{r}_{01} = (2, 0, 0), \quad \vec{r}_{02} = (1, 1, 1) \Rightarrow \vec{r}_{01} - \vec{r}_{02} = (1, -1, -1)$ $\vec{u}_1 \times \vec{u}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} \\ 1 & 1 \\ 0 & -1 \end{vmatrix} = (2, -2, -1)$ $d = \frac{ (\vec{r}_{01} - \vec{r}_{02}) \cdot (\vec{u}_1 \times \vec{u}_2) }{\ \vec{u}_1 \times \vec{u}_2\ } = \frac{ (1, -1, -1) \cdot (2, -2, -1) }{\sqrt{4 + 4 + 1}}$ $= \frac{ 2 + 2 + 1 }{\sqrt{9}} = \frac{5}{3}$ $\therefore d = \frac{5}{3}$

Reading: Nelson Textbook, Pages 534-539

Homework: Nelson Textbook: Page 540 #1a, 2a, 3a, 5abc, 6a, 7a, 8, 10