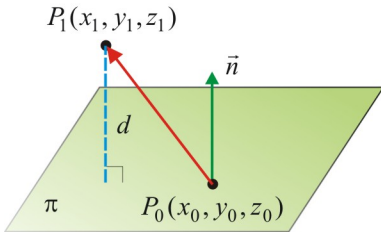


**9.6 Distance from a Point to a Plane**

**A Distance from a Point to a Plane**

Let consider a plane  $\pi$  with a *normal vector*  $\vec{n}$  and a point  $P_0(x_0, y_0, z_0)$  on this plane. The *distance* from a point  $P_1(x_1, y_1, z_1)$  to the plane  $\pi$  is given by the *scalar projection* of the vector  $\vec{P_0P_1}$  onto the normal vector  $\vec{n}$  :

$$d = \frac{|\vec{P_0P_1} \cdot \vec{n}|}{\|\vec{n}\|} \quad (1)$$



Ex 1. For each case, find the distance between the given plane and the given point.

a)  $\vec{r} = (1,0,2) + t(0,1,2) + s(2,0,1)$  ,  $B(2,3,0)$

$$\vec{P_0B} = (2,3,0) - (1,0,2) = (1,3,-2)$$

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = (1,4,-2)$$

$$\therefore d = \frac{|\vec{P_0B} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|1+12+4|}{\sqrt{1+16+4}} = \frac{17}{\sqrt{21}}$$

b)  $\begin{cases} x = 1 - t + s \\ y = 2 - t - 2s \\ z = -1 + 2t - 3s \end{cases}$   $M(1,0,-2)$

$$\vec{P_0M} = (1,0,-2) - (1,2,-1) = (0,-2,-1)$$

$$\vec{u} = (-1,-1,2); \quad \vec{v} = (1,-2,-3)$$

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -1 & 2 \\ 1 & -2 & -3 \end{vmatrix} = (7,-1,3)$$

$$\therefore d = \frac{|\vec{P_0M} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|0+2-3|}{\sqrt{49+1+9}} = \frac{1}{\sqrt{59}}$$

**B Distance from a Point to a Plane (II)**

If the plane  $\pi$  is given by the *Cartesian equation*  $\pi : Ax + By + Cz + D = 0$  , then the *distance* from a point  $P_1(x_1, y_1, z_1)$  to the plane is given by:

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}} \quad (2)$$

Indeed,

$$P_0 \in \pi \Rightarrow Ax_0 + By_0 + Cz_0 + D = 0$$

$$\vec{P_0P_1} \cdot \vec{n} = (x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot (A, B, C)$$

$$= Ax_1 + By_1 + Cz_1 - Ax_0 - By_0 - Cz_0 = Ax_1 + By_1 + Cz_1 + D$$

Ex 2. Find the distance between the point  $R(-2,0,3)$  and the plane  $\pi : 2x - 3y + z - 6 = 0$  .

$$\therefore d = \frac{|2(-2) - 3(0) + 3 - 6|}{\sqrt{2^2 + (-3)^2 + 1^2}} = \frac{|-4 + 3 - 6|}{\sqrt{4+9+1}} = \frac{7}{\sqrt{14}} = \frac{\sqrt{14}}{2}$$

**C Distance between two Parallel Planes**

To get the *distance* between *two parallel planes*:

- Find a specific point into one of these planes.
- Find the distance between that specific point and the other plane using one of the formulas above.

Ex 3. Find the distance between the parallel planes.

$$\pi_1 : 3x + 6y - 9z - 3 = 0, \quad \pi_2 : 2x + 4y - 6z - 4 = 0$$

$$P_1(1,0,0) \in \pi_1$$

$$\therefore d = \frac{|2(1) + 4(0) - 6(0) - 4|}{\sqrt{4+16+36}} = \frac{2}{\sqrt{56}} = \frac{1}{\sqrt{14}}$$

Ex 4. Consider a plane  $\pi$  with the x-, y-, and z- intercepts equal to  $a$ ,  $b$ , and  $c$  respectively.

a) Find the Cartesian equation of the plane  $\pi$ .

$$Ax + By + Cz + D = 0$$

$$x\text{-int} = -\frac{D}{A} = a \Rightarrow A = \frac{-D}{a}$$

$$y\text{-int} = -\frac{D}{B} = b \Rightarrow B = \frac{-D}{b}$$

$$z\text{-int} = -\frac{D}{C} = c \Rightarrow C = \frac{-D}{c}$$

$$\frac{-D}{a}x + \frac{-D}{b}y + \frac{-D}{c}z + D = 0$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0$$

b) Find the distance from the origin to the plane  $\pi$ .

$$d = \frac{|\frac{0}{a} + \frac{0}{b} + \frac{0}{c} - 1|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$

Ex 5. Consider the plane  $\pi: -x + 2y - 3z + 10 = 0$  and the point  $P(2,3,0)$ .

a) Find the vector equation of the line  $L_{\perp}$  that passes through the point  $P$  and is perpendicular to the plane  $\pi$ .

The line  $L_{\perp}$  is perpendicular to the plane  $\pi$  and therefore is parallel to the normal vector  $\vec{n} = (-1, 2, -3)$  to the plane  $\pi$ .

The normal vector  $\vec{n} = (-1, 2, -3)$  may be used as direction vector for the line  $L_{\perp}$ . The vector equation of the line  $L_{\perp}$  is:

$$L_{\perp}: \vec{r} = (2, 3, 0) + t(-1, 2, -3), \quad t \in \mathbb{R}$$

b) Find the point of intersection  $F$  between the perpendicular line  $L_{\perp}$  and the plane  $\pi$ .

$$L_{\perp}: \begin{cases} x = 2 - t \\ y = 3 + 2t \\ z = -3t \end{cases}$$

$$-(2 - t) + 2(3 + 2t) - 3(-3t) + 10 = 0$$

$$14t = -14 \Rightarrow t = -1 \Rightarrow F: \begin{cases} x = 2 - (-1) = 3 \\ y = 3 + 2(-1) = 1 \\ z = -3(-1) = 3 \end{cases}$$

$$\therefore F(3, 1, 3)$$

c) Find the shortest distance between the point  $P$  and the plane  $\pi$ .

$$d = \|\overline{PF}\| = \|(1, -2, 3)\| = \sqrt{1 + 4 + 9} = \sqrt{14}$$

**Reading:** Nelson Textbook, Pages 542-549

**Homework:** Nelson Textbook: Page 549 # 1, 2, 3, 5, 6